PRINCIPLES OF ANALYSIS
LECTURE 24 - FUNDAMENTAL THEOREM OF CALCULUS

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Proposition 1. *Let $f : [a,b] \to \mathbb{R}$ be differentiable on $[a,b]$ such that $f' : [a,b] \to \mathbb{R}$ is integrable on $[a,b]$. Then*

$$\int_a^b f'\,dx = f(b) - f(a).$$

Proof. Let $P = \{x_0, \dots, x_n\}$ be any partition of $[a,b]$. By the Mean Value Theorem, there exists $c_i \in [x_{i-1}, x_i]$ such that $f'(c_i) = \frac{f(x_i)-f(x_{i-1})}{x_i - x_{i-1}}$. Thus,

$$m_f(P,i) \leq \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} \leq M_f(P,i),$$

for $i = 1, \dots, n$. Therefore,

$$\sum_{i=1}^n m_f(P,i)(x_i - x_{i-1}) \leq \sum_{i=1}^n \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}(x_i - x_{i-1}) \leq \sum_{i=1}^n M_f(P,i)(x_i - x_{i-1}).$$

Now

$$\sum_{i=1}^n \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}(x_i - x_{i-1}) = \sum_{i=1}^n (f(x_i) - f(x_{i-1})) = f(b) - f(a).$$

Thus

$$L_f(P) \leq f(b) - f(a) \leq U_f(P).$$

Since this is true for every partition,

$$\underline{\int_a^b} f'\,dx \leq f(b) - f(a) \leq \overline{\int_a^b} f'\,dx.$$

Since f' is integrable, the upper sum equals the lower sum, so

$$\int_a^b f'\,dx = f(b) - f(a).$$

□

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Proposition 2. *Let $f : [a, b] \to \mathbb{R}$ be integrable. If $f(x) \geq 0$ for all $x \in [a, b]$, then $\int_a^b f\,dx \geq 0$.*

Proof. Clearly for any partition P, we have $U_f(P) \geq 0$. Thus taking the infimum gives $\overline{\int} f\,dx \geq 0$. But f is integrable, so $\int f\,dx = \overline{\int} f\,dx \geq 0$. □

Proposition 3. *Let $f, g : [a, b] \to \mathbb{R}$ be integrable. If $f(x) \leq g(x)$ for every $x \in [a, b]$, then*

$$\int_a^b f\,dx \leq \int_a^b g\,dx.$$

Proof. We see that $(g - f)(x) \geq 0$ for every $x \in [a, b]$, so

$$\int_a^b g\,dx - \int_a^b f\,dx = \int_a^b (g - f)\,dx \geq 0,$$

which implies that

$$\int_a^b f\,dx \leq \int_a^b g\,dx.$$

□

Proposition 4. *Let $f(x) = M$ be a constant. Then $\int_a^b f\,dx = M(b - a)$.*

Proof. Let $F(x) = Mx$. Then $f(x) = F'(x)$, and $\int_a^b f\,dx = F(b) - F(a) = Mb - Ma = M(b - a)$. □

Proposition 5. *Let $f : [a, b] \to \mathbb{R}$ be integrable and bounded, so that $|f(x)| \leq M$ for some $M > 0$. Then $\int_a^b f\,dx \leq M(b - a)$.*

Let $D \subset \mathbb{R}$ and let $f : D \to \mathbb{R}$ be any function. Define a function $f^+ : D \to \mathbb{R}$ by

$$f^+(x) = \begin{cases} f(x) & \text{if } f(x) \geq 0; \\ 0 & \text{otherwise.} \end{cases}$$

Also define $f^- : D \to \mathbb{R}$ by

$$f^-(x) = \begin{cases} -f(x) & \text{if } f(x) \leq 0; \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

(a) $f^- = (-f)^+$;
(b) $f = f^+ - f^-$;
(c) $|f| = f^+ + f^-$.

Proposition 6. *Let $f : [a, b] \to \mathbb{R}$ be integrable. Then $f^+ : [a, b] \to \mathbb{R}$ is also integrable.*

Proof. Let $\epsilon > 0$ and let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a, b]$ such that $U_f(P) - L_f(P) < \epsilon$. Then for every i we have $M_f(P, i) \geq M_{f^+}(P, i)$, and $m_f(P, i) \leq m_{f^+}(P, i)$; this implies that $M_{f^+}(P, i) - m_{f^+}(P, i) \leq M_f(P, i) - m_f(P, i)$. Thus

$$\begin{aligned} U_{f^+}(P) - L_{f^+}(P) &= \sum_{i=1}^{n} (M_{f^+}(P, i) - m_{f^+}(P, i))(x_i - x_{i-1}) \\ &\leq \sum_{i=1}^{n} (M_f(P, i) - m_f(P, i))(x_i - x_{i-1}) \\ &= U_f(P) - L_f(P) \\ &< \epsilon. \end{aligned}$$

This shows that f^+ is integrable. □

Proposition 7. *Let $f : [a, b] \to \mathbb{R}$ be integrable. Then $|f|$ is integrable, and $|\int_a^b f\, dx| \leq \int_a^b |f|\, dx$.*

Proof. We just saw that f^+ is integrable. Also $-f$ is integrable, so $f^- = (f)^+$ is integrable. Therefore $|f| = f^+ + f^-$ is integrable.

$$\Big|\int_a^b f\, dx\Big| = \Big|\int_a^b f^+\, dx - \int_a^b f^-\, dx\Big| \leq \Big|\int_a^b f^+\, dx + \int_a^b f^-\, dx\Big| = \int_a^b |f|\, dx.$$

□

Observation 1. As a convenience, if $a > b$, then define $\int_a^b f\,dx = -\int_b^a f\,dx$. Then it follows from a previous proposition that

$$\int_a^b f\,dx - \int_a^c f\,dx = \int_c^b f\,dx.$$

Observation 2. If c is constant, then $c = \frac{\int_a^b c\,dx}{b-a}$.

Proposition 8. *Let $f : [a,b] \to \mathbb{R}$ be bounded and integrable. Define*

$$F : [a,b] \to \mathbb{R} \quad \text{by} \quad F(x) = \int_a^x f(t)\,dt.$$

Then

- **(a)** *F is uniformly continuous on $[a,b]$;*
- **(b)** *if f is continuous at x_0, then F is differentiable at x_0 and $F'(x_0) = f(x_0)$.*

Proof. Since f is bounded, there exits $M > 0$ such that $|f(x)| \leq M$ for all $x \in [a,b]$.

Let $\epsilon > 0$ and let $\delta = \epsilon/M$. Let $x, y \in [a,b]$, and suppose that $|x - y| < \delta$. Then

$$\begin{aligned}|F(x) - F(y)| &= |\int_a^x f(t)\,dt - \int_a^y f(t)\,dt| \\ &= |\int_x^y f(t)\,dt| \\ &\leq M|x-y| \\ &= \epsilon.\end{aligned}$$

Therefore, F is continuous.

Suppose that f is continuous at x_0. Select $\epsilon > 0$; then there exists $\delta > 0$ such that if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \frac{\epsilon}{2}$. Note that this says that for every $t \in [x, x_0]$, we have $|f(t) - f(x_0)| < \frac{\epsilon}{2}$. Compute

$$\begin{aligned}\Big|\frac{F(x) - F(x_0)}{x - x_0} - f(x_0)\Big| &= \Big|\frac{\int_a^x f(t)\,dt - \int_a^{x_0} f(t)\,dt}{x - x_0} - f(x_0)\Big| \\ &= \Big|\frac{\int_{x_0}^x f(t)\,dt}{x - x_0} - \frac{\int_{x_0}^x f(x_0)\,dx}{x - x_0}\Big| \\ &= \Big|\frac{\int_{x_0}^x (f(t) - f(x_0))\,dt}{x - x_0}\Big| \\ &\leq \Big|\frac{(\epsilon/2)(x - x_0)}{x - x_0}\Big| \\ &= \frac{\epsilon}{2} \\ &< \epsilon.\end{aligned}$$

□

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